follows from (2), (12) and the integration by parts formula. Let points of an elastic solid receive a small additional displacement $\mathbf{w}_1$ under the effect of the additional forces $\mathbf{k}_1$ and $\mathbf{f}_1'$, and the displacement $\mathbf{w}_2$ under the effect of the system of forces $\mathbf{k}_2$, $\mathbf{f}_2'$. A theorem on reciprocity of work of the additional forces on the additional displacements follows from (2), (9)

$$\iiint_{V^\circ} \rho \mathbf{k}_1 \cdot \mathbf{w}_2 \, d\tau + \iint_O \mathbf{f}_1' \cdot \mathbf{w}_2 \, dO = \iiint_{V^\circ} \rho \mathbf{k}_2 \cdot \mathbf{w}_1 \, d\tau + \iint_O \mathbf{f}_2' \cdot \mathbf{w}_1 \, dO \qquad (18)$$

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ON THE THEORY OF ELASTICITY OF INHOMOGENEOUS MEDIA

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The general solution is obtained of the equilibrium equations in displacements for inhomogeneous isotropic media, whose elastic characteristics are differentiable functions of the Cartesian coordinates. It is shown that the components of the displacement vector in the three-dimensional problem of elasticity theory can always be expressed in terms of two functions which satisfy second and fourth order linear partial differential equations, respectively.

Of the earlier research devoted to analogous problems, the paper [1] should first be noted in which an equation is derived for the Airy stress function in the two-dimensional problem of the theory of elasticity of an inhomogeneous medium. A general solution of the equilibrium equations in displacements is obtained in [2, 3] for the case of axisymmetric deformation of bodies of revolution whose elastic moduli vary exponentially as a function of the coordinate z. A power-law change in the elastic modulus was investigated in [4] with primary attention paid to the plane problem. General solutions of the equilibrium equations in displacements for the three-dimensional case and an arbitrary law of variation of the elastic characteristics of an inhomogeneous medium has apparently not been examined.

1. Let us consider the shear modulus G of an inhomogeneous medium and the Poisson ratio ν to be differentiable functions of the Cartesian coordinate z. The equilibrium equations in displacements in the absence of mass forces have the form

$$\begin{gathered}\frac{G}{1-2\nu}\frac{\partial\theta}{\partial x}+G\nabla^2 u_x+\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right)\frac{dG}{dz}=0\\ \frac{G}{1-2\nu}\frac{\partial\theta}{\partial y}+G\nabla^2 u_y+\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right)\frac{dG}{dz}=0\\ \frac{G}{1-2\nu}\frac{\partial\theta}{\partial z}+G\nabla^2 u_z+\theta\frac{d\lambda}{dz}+2\frac{\partial u_z}{\partial z}\frac{dG}{dz}=0\qquad\left(\lambda=\frac{2G\nu}{1-2\nu}\right)\end{gathered}\tag{1.1}$$

Here u_x, u_y, u_z are components of the displacement vector, θ is the relative volume deformation. Let us differentiate the first equation of the system (1.1) with respect to y, and the second with respect to x and let us subtract. We hence find that

$$\partial u_x/\partial y-\partial u_y/\partial x=\chi\tag{1.2}$$

$\chi\ (x,\ y,\ z)$ is an arbitrary function satisfying the equation

$$\nabla^2\chi+q(z)\frac{\partial\chi}{\partial z}=0,\qquad q(z)=\frac{d}{dz}\ln G\tag{1.3}$$

Here u_x^+, u_y^+ is a particular solution of the inhomogeneous equation (1.2), and u_x°, u_y° is the general solution of its corresponding homogeneous equation. It follows from the homogeneous equation that there exists a function $F\ (x,\ y,\ z)$ such that

$$u_x^\circ=\partial F/\partial x,\qquad u_y^\circ=\partial F/\partial y\tag{1.4}$$

To seek the particular solution, let us introduce a new function $N\ (x,\ y,\ z)$, by assuming

$$\chi=\partial^2 N/\partial x^2+\partial^2 N/\partial y^2\tag{1.5}$$

Substituting this expression into (1.2), we obtain that the particular solution can be taken in the form

$$u_x^+=\partial N/\partial y,\qquad u_y^+=-\partial N/\partial x\tag{1.6}$$

Therefore, the displacements u_x and u_y can be expressed in terms of two auxiliary functions F and N in the most general case. Substituting the general solution of the inhomogeneous equation (1.2), in conformity with (1.4), (1.6), into (1.1), we obtain a system of three differential equations for F, N and u_z :

$$\partial Q_1/\partial x+\partial Q_2/\partial y=0,\qquad \partial Q_1/\partial y-\partial Q_2/\partial x=0\tag{1.7}$$

$$\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\left[G\left(\frac{\partial F}{\partial z}+u_z\right)\right]+\frac{\partial}{\partial z}\left\{\frac{2G}{1-2\nu}\left[\nu\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)F+(1-\nu)\frac{\partial u_z}{\partial z}\right]\right\}=0$$

$$Q_1=\frac{2G}{1-2\nu}\left[(1-\nu)\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)F+\nu\frac{\partial u_z}{\partial z}\right]+\frac{\partial}{\partial z}\left[G\left(\frac{\partial F}{\partial z}+u_z\right)\right]$$

$$Q_2=G\nabla^2 N+\frac{\partial N}{\partial z}\frac{dG}{dz}$$

It follows from the first two equations of the system (1.7) that the functions Q_1 and Q_2 are interrelated by the Cauchy-Riemann conditions, and can hence be represented as

$$Q_1 = \partial\omega/\partial x, \qquad Q_2 = \partial\omega/\partial y \tag{1.8}$$

where $\omega\ (x,\ y,\ z)$ is an arbitrary harmonic function satisfying the two-dimensional Laplace equation (the variable z plays the part of a parameter). The system of differential equations (1.7) hence decomposes into two independent subsystems, one of which determines the functions F and u_z

$$\begin{gathered}\frac{2G}{1-2\nu}\left[(1-\nu)\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)F+\nu\,\frac{\partial u_z}{\partial z}\right]+\frac{\partial}{\partial z}\left[G\left(\frac{\partial F}{\partial z}+u_z\right)\right]=\frac{\partial\omega}{\partial x}\\ \left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)\left[G\left(\frac{\partial F}{\partial z}+u_z\right)\right]+\frac{\partial}{\partial z}\left\{\frac{2G}{1-2\nu}\left[\nu\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)F+(1-\nu)\frac{\partial u_z}{\partial z}\right]\right\}=0\end{gathered} \tag{1.9}$$

and the other, the function N

$$G\nabla^2 N+\frac{\partial N}{\partial z}\,\frac{dG}{dz}=\frac{\partial\omega}{\partial y} \tag{1.10}$$

This last equation evidently does not contradict conditions (1.5) and (1.3).

Now, let us show that the harmonic function $\omega\ (x,\ y,\ z)$ can be taken equal to zero without limiting the generality. Let F^+, u_z^+ and N^+ denote particular solutions of the inhomogeneous differential equations (1.9) and (1.10), and F°, $\overset{\circ}{u_z}$ and N° the general solutions of their corresponding homogeneous equations. By direct substitution it can be seen that

$$F^+=\int\frac{dz}{G}\int\frac{\partial\omega}{\partial x}\,dz, \qquad u_z^+=0; \qquad N^+=\int\frac{dz}{G}\int\frac{\partial\omega}{\partial y}\,dz \tag{1.11}$$

can be taken as the particular solution of the system (1.9) and of (1.10).
Utilizing (1.11), we have

$$u_x=\partial F^\circ/\partial x+\partial N^\circ/\partial y, \qquad u_y=\partial F^\circ/\partial y-\partial N^\circ/\partial x \tag{1.12}$$

Moreover, it follows from the condition $u_z^+=0$ that $u_z=u_z^\circ$. Therefore, the displacements u_x, u_y, u_z depend only on the functions F°, u_z° and N° which are solutions of the system (1.9) and of (1.10) if we put $\omega=0$ therein. Therefore, the function ω cannot exert any influence on the components of the displacement vector, and it can be discarded without loss of generality. Let us put $\omega=0$ in (1.9) and (1.10). We then obtain a second order equation for N

$$\nabla^2 N+q\,(z)\,\frac{\partial N}{\partial z}=0 \tag{1.13}$$

which goes over into the three-dimensional Laplace equation in the case of a homogeneous medium. We obtain a homogeneous system of differential equations for F and u_z which after introducing the new unknowns

$$\sigma=\frac{2G}{1-2\nu}\left[\nu\left(\nabla^2-\frac{\partial^2}{\partial z^2}\right)F+(1-\nu)\,\frac{\partial u_z}{\partial z}\right], \qquad \tau=G\left(\frac{\partial F}{\partial z}+u_z\right) \tag{1.14}$$

becomes

$$\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\tau + \frac{\partial \sigma}{\partial z} = 0, \qquad \left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right) F + \frac{1-\nu}{2G}\frac{\partial \tau}{\partial z} + \frac{\nu}{2G}\sigma = 0$$
$$\frac{\partial F}{\partial z} - \frac{\tau}{G} + u_z = 0, \qquad \frac{2G}{1-2\nu}\left[\nu\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right) F + (1-\nu)\frac{\partial u_z}{\partial z}\right] - \sigma = 0 \tag{1.15}$$

Let us now introduce the auxiliary function $L\,(x,\,y,\,z)$ by assuming

$$\sigma = \left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)^2 L, \qquad \tau = -\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial z}$$
$$F = -\frac{1}{2G}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right) L \tag{1.16}$$

Substituting these expressions into the first two equations of the system (1.15), we will see that they are satisfied for any selection of the function L. Substitution into the third equation yields

$$u_z = -\frac{1}{G}\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial z} + \frac{\partial}{\partial z}\left[\frac{1}{2G}\left(\nu\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right] \tag{1.17}$$

There remains to select the function L in such a way that the last equation of the system (1.15) would be satisfied. Substituting (1.16) and (1.17) into this equation, we obtain a fourth order equation for L

$$\nabla^2\nabla^2 L - \frac{G}{1-\nu}\left\{\frac{1}{G}\left[\frac{\partial^2}{\partial z^2}(\nu\nabla^2 L) - \nu\frac{\partial^2}{\partial z^2}\nabla^2 L\right] - 2\frac{\partial}{\partial z}\left[(1-\nu)\nabla^2 L\right]\frac{d}{dz}\left(\frac{1}{G}\right) + \right.$$
$$\left. + \left(\nu\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\frac{d^2}{dz^2}\left(\frac{1}{G}\right)\right\} = 0 \tag{1.18}$$

This equation is somewhat simplified if the Poisson's ratio is $\nu = \text{const}$, and it can be written as

$$\nabla^2\left(\frac{1}{G}\,\nabla^2 L\right) - \frac{1}{1-\nu}\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right) L\,\frac{d^2}{dz^2}\left(\frac{1}{G}\right) = 0 \tag{1.19}$$

Let us note that when the medium is homogeneous, (1.18) reduces to the biharmonic equation.

Therefore, the solution of the system (1.1) in the most general case is successfully reduced to the solution of two linear differential equations, one of second order, and the other of fourth order. Taking into account (1.16) and (1.17), let us write the general solution of the equilibrium equations in displacements for inhomogeneous media in the following final form

$$u_x = -\frac{1}{2G}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial x} + \frac{\partial N}{\partial y}, \qquad u_y = -\frac{1}{2G}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial y} - \frac{\partial N}{\partial x}$$
$$u_z = -\frac{1}{G}\left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\frac{\partial L}{\partial z} + \frac{\partial}{\partial z}\left[\frac{1}{2G}\left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right) L\right] \tag{1.20}$$

Here N is the function satisfying (1.13), and L satisfies (1.18). From (1.20) we find the components of the stress tensor

$$\sigma_x = \left(\nu \frac{\partial^2}{\partial y^2} \nabla^2 + \frac{\partial^4}{\partial x^2 \partial z^2}\right) L + 2G \frac{\partial^2 N}{\partial x \partial y}$$

$$\sigma_y = \left(\nu \frac{\partial^2}{\partial x^2} \nabla^2 + \frac{\partial^4}{\partial y^2 \partial z^2}\right) L - 2G \frac{\partial^2 N}{\partial x \partial y}, \qquad \sigma_z = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 L$$

$$\tau_{xy} = -\left(\nu \nabla^2 - \frac{\partial^2}{\partial z^2}\right) \frac{\partial^2 L}{\partial x \partial y} - G \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) N \tag{1.21}$$

$$\tau_{zx} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{\partial^2 L}{\partial x \partial z} + G \frac{\partial^2 N}{\partial y \partial z}, \quad \tau_{zy} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \frac{\partial^2 L}{\partial y \partial z} - G \frac{\partial^2 N}{\partial x \partial z}$$

The components of the displacement vector and the stress tensor are written in cylindrical coordinates r, β, z as follows:

$$u_r = -\frac{1}{2G} \frac{\partial}{\partial r} \left(\nu \nabla^2 - \frac{\partial^2}{\partial z^2}\right) L + \frac{1}{r} \frac{\partial N}{\partial \beta}$$

$$u_\beta = -\frac{1}{2Gr} \frac{\partial}{\partial \beta} \left(\nu \nabla^2 - \frac{\partial^2}{\partial z^2}\right) L - \frac{\partial N}{\partial r} \tag{1.22}$$

$$u_z = -\frac{1}{G} \left(\nabla^2 - \frac{\partial^2}{\partial z^2}\right) \frac{\partial L}{\partial z} + \frac{\partial}{\partial z} \left[\frac{1}{2G} \left(\nu \nabla^2 - \frac{\partial^2}{\partial z^2}\right) L\right]$$

$$\sigma_r = \frac{\nu}{r} \left(\frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial^2}{\partial \beta^2}\right) \nabla^2 L + \frac{\partial^4 L}{\partial r^2 \partial z^2} + \frac{2G}{r} \frac{\partial}{\partial \beta} \left(\frac{\partial N}{\partial r} - \frac{N}{r}\right)$$

$$\sigma_\beta = \nu \frac{\partial^2}{\partial r^2} \nabla^2 L + \frac{1}{r} \left(\frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial^2}{\partial \beta^2}\right) \frac{\partial^2 L}{\partial z^2} - \frac{2G}{r} \frac{\partial}{\partial \beta} \left(\frac{\partial N}{\partial r} - \frac{N}{r}\right)$$

$$\sigma_z = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \beta^2}\right)^2 L$$

$$\tau_{r\beta} = -\frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) \left(\nu \nabla^2 - \frac{\partial^2}{\partial z^2}\right) L - G \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \beta^2}\right) N \tag{1.23}$$

$$\tau_{\beta z} = -\frac{1}{r} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \beta^2}\right) \frac{\partial^2 L}{\partial \beta \partial z} - G \frac{\partial^2 N}{\partial r \partial z}$$

$$\tau_{rz} = -\frac{\partial^2}{\partial r \partial z} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \beta^2}\right) L + \frac{G}{r} \frac{\partial^2 N}{\partial \beta \partial z}$$

If the medium is homogeneous, then the general solution of the system (1.1) is represented by a set of harmonic N and biharmonic L functions. It can hence be shown that the known constructions of the general solutions of the Lamé equations obtained by P.F. Papkovich, B.G. Galerkin, E. Trefftz and other authors are particular cases of that considered above. Thus, for example, we arrive at the Papkovich solution if we set

$$L = -\int dz \int dz \int dz \int \left[\left(\nu \nabla^2 + \frac{\partial^2}{\partial z^2}\right) \Phi + 4(1-\nu) \left(\frac{\partial \Phi_1}{\partial x} + \frac{\partial \Phi_2}{\partial y}\right)\right] dz$$

$$N = -\frac{4(1-\nu)}{2G} \int dz \int \left(\frac{\partial \Phi_1}{\partial y} - \frac{\partial \Phi_2}{\partial x}\right) dz \qquad (\Phi = \Phi_0 + x\Phi_1 + y\Phi_2 + z\Phi_3) \tag{1.24}$$

Here Φ_0, Φ_1, Φ_2, Φ_3 are arbitrary harmonic functions. Substituting (1.24) into (1.20), we obtain

$$2G\,(u_x, u_y, u_z) = 4\,(1-\nu)\,\Phi_{1,2,3} - \frac{\partial\Phi}{\partial x, y, z} \tag{1.25}$$

which is equivalent to the solution presented in [5].

2. It is possible to go over to the two-dimensional problem of the theory of elasticity of an inhomogeneous medium if it is assumed that $N = 0$ and L is independent of the coordinates x or y in (1.18) and (1.20). For example, let $L = L\,(x, z)$. Then according to (1.18), the function L must satisfy the equation

$$\Delta\Delta L - \frac{G}{1-\nu^*}\left\{\frac{1}{G}\left[\frac{\partial^2}{\partial z^2}(\nu^*\Delta L) - \nu^*\frac{\partial^2}{\partial z^2}\Delta L\right] - 2\frac{\partial}{\partial z}\left[(1-\nu^*)\,\Delta L\right]\frac{d}{dz}\left(\frac{1}{G}\right) + \right.$$
$$\left. + \left[\nu^*\frac{\partial^2 L}{\partial x^2} - (1-\nu^*)\frac{\partial^2 L}{\partial z^2}\right]\frac{d^2}{dz^2}\left(\frac{1}{G}\right)\right\} = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \tag{2.1}$$

Here $\nu^* = \nu$ for the case of deformation occuring in a plane parallel to xOz and $\nu^* = \nu(1+\nu)$ for the case of the generalized plane stress state. If the Poisson ratio is constant, then (2.1) can be written as

$$\Delta\left(\frac{1}{G}\,\Delta L\right) - \frac{1}{1-\nu^*}\frac{\partial^2 L}{\partial x^2}\frac{d^2}{dz^2}\left(\frac{1}{G}\right) = 0 \tag{2.2}$$

Expressions to determine the displacements follow from (1.20):

$$u_x = -\frac{1}{2G}\left[\nu^*\frac{\partial^2}{\partial x^2} - (1-\nu^*)\frac{\partial^2}{\partial z^2}\right]\frac{\partial L}{\partial x} \tag{2.3}$$
$$u_z = -\frac{1}{G}\frac{\partial^3 L}{\partial x^2\,\partial z} + \frac{\partial}{\partial z}\left\{\frac{1}{2G}\left[\nu^*\frac{\partial^2 L}{\partial x^2} - (1-\nu^*)\frac{\partial^2 L}{\partial z^2}\right]\right\}$$

The stresses are determined by means of the formulas

$$\sigma_x = \frac{\partial^4 L}{\partial x^2\,\partial z^2}, \qquad \sigma_y = \frac{\partial^4 L}{\partial x^4}, \qquad \tau_{xy} = -\frac{\partial^4 L}{\partial x^3\,\partial z} \tag{2.4}$$

If the function $\Lambda = \partial^2 L/\partial x^2$ is introduced, then it is easy to see that it is an analog of the Airy stress function.

3. Let us apply the method of separation of variables to seek the particular solutions of (1.13) and (1.18). Let us assume

$$N\,(x, y, z) = \psi_1\,(x, y)\,\varphi_1\,(z), \qquad L\,(x, y, z) = \psi_2\,(x, y)\,\varphi_2\,(z) \tag{3.1}$$

Substituting (3.1) into (1.13) and (1.18), we see that the variables separate, if the functions $\psi_1\,(x, y)$ and $\psi_2\,(x, y)$ satisfy the Helmholtz equation

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 + \alpha^2\psi = 0 \tag{3.2}$$

where α is an arbitrary numerical parameter. For the functions $\varphi_1\,(z)$ and $\varphi_2\,(z)$ we obtain

$$\frac{d^2\varphi_1}{dz^2} + q\,(z)\frac{d\varphi_1}{dz} - \alpha^2\,\varphi_1 = 0 \tag{3.3}$$

$$\frac{d^2}{dz^2}\left\{\frac{1}{G}\left[(1-\nu)\frac{d^2\varphi_2}{dz^2} + \alpha^2\nu\varphi_2\right]\right\} - 2\alpha^2\frac{d}{dz}\left(\frac{1}{G}\frac{d\varphi_2}{dz}\right) +$$
$$+ \frac{\alpha^2\nu}{G}\frac{d^2\varphi_2}{dz^2} + \frac{\alpha^4\,(1-\nu)}{G}\,\varphi_2 = 0 \tag{3.4}$$

If the Poisson ratio is $\nu =$ const, this last equation simplifies somewhat

$$\frac{d^4\varphi_2}{dz^4} - 2q(z)\frac{d^3\varphi_2}{dz^3} + \left[q^2(z) - q'(z) - 2\alpha^2\right]\frac{d^2\varphi_2}{dz^2} + 2\alpha^2 q(z)\frac{d\varphi_2}{dz} + $$
$$+ \alpha^2\left\{\alpha^2 + \frac{\nu}{1-\nu}\left[q^2(z) - q'(z)\right]\right\}\varphi_2 = 0, \quad q'(z) = \frac{d}{dz}q(z) \qquad (3.5)$$

The equation of vibrations in a plane (3.2) admits the following particular solutions [6], which can be utilized in many problems of the theory of elasticity: in Cartesian coordinates

$$\psi = e^{i(m_1x+m_2y)}, \qquad m_1^2 + m_2^2 = \alpha^2 \qquad (3.6)$$

$$\psi = (A + Bx)e^{i\alpha y} + (C + Dy)e^{i\alpha x} \qquad (3.7)$$

in cylindrical coordinates

$$\psi = e^{\pm im\beta}\left[AJ_m(\alpha r) + BY_m(\alpha r)\right] \qquad (m = 0, 1, 2\ldots) \qquad (3.8)$$

$$\psi = \beta\left[CJ_0(\alpha r) + DY_0(\alpha r)\right] \qquad (r = \sqrt{x^2+y^2}, \ \mathrm{tg}\,\beta = y/x) \qquad (3.9)$$

Here A, B, C, D are arbitrary constants, $J_m(\alpha r)$ and $Y_m(\alpha r)$ are Bessel functions of the first and second kind of order m, respectively.

Therefore, the fundamental difficulty is to find φ_1 and φ_2 from the differential equations (3.3) and (3.4), whose solutions are successfully expressed in terms of known functions only in the simplest cases for a given law of variation of the elastic characteristics.

4. Let us examine some of these cases. If the elastic modulus varies exponentially, and the Poisson ratio is $\nu = \mathrm{const}$, then (3.3) and (3.4) transform into linear differential equations with constant coefficients, and their solution is not difficult. Individual problems referring to this case have already been considered. Thus, an equation analogous to (3.5) has been obtained in [2] in the investigation of the axisymmetric deformation of bodies of revolution for an elastic modulus of the form $E(z) = E_0 e^{\gamma z}$. If we put $E(z) = k(z+h)^b$ in (3.3), then $q(z) = b/(z+h)$ and we arrive at an equation equivalent to that examined in [7]. A power-law variation in the modulus has been considered in the plane problem of the theory of elasticity of an inhomogeneous medium in [4], where an equation analogous to (3.6) has been investigated for $q(z) = b/(z+h)$. However, not all of its solutions were examined therein, hence we consider this case in more detail.

Putting $E(z) = k(z+h)^b$ in (3.5), and making the change of variable $z_1 = z + h$, we obtain

$$\frac{d^4\varphi_2}{dz_1^4} - \frac{2b}{z_1}\frac{d^3\varphi_2}{dz_1^3} + \left[\frac{b(b+1)}{z_1^2} - 2\alpha^2\right]\frac{d^2\varphi_2}{dz_1^2} + \frac{2\alpha^2 b}{z_1}\frac{d\varphi_2}{dz_1} + \alpha^2\left(\alpha^2 + \frac{\nu}{1-\nu}b\frac{b+1}{z_1^2}\right)\varphi_2 = 0 \qquad (4.1)$$

According to [4], the solution of this equation is

$$\varphi_2 = z_1^{1/2(b+1)}\left\{C_1M_{\lambda,\mu}(2\alpha z_1) + C_2W_{\lambda,\mu}(2\alpha z_1) + C_3M_{-\lambda,\mu}(2\alpha z_1) + C_4W_{-\lambda,\mu}(2\alpha z_1)\right\}$$

$$\lambda = 1/2\sqrt{(b+1)\left[1 - \nu b/(1-\nu)\right]}, \quad \mu = -1 - 1/2\,b \qquad (4.2)$$

Here $M_{\lambda,\mu}(2\alpha z_1)$, $W_{\lambda,\mu}(2\alpha z_1)$ are Whittaker functions, and the C_1, C_2, C_3, C_4 are arbitrary constants. However, if the Poisson ratio ν is connected with the exponent b by the dependence $\nu = 1/(b+1)$ or $b = -1$, then $\lambda = 0$ and (4.2) yields only two independent solutions of (4.1). To seek the general solution in this case, let us put $\eta = \alpha z_1$ in (4.1), and let us write it as

$$\left[\frac{d^2}{d\eta^2}+\frac{1-b}{\eta}\frac{d}{d\eta}-\left(1+\frac{1+b}{\eta^2}\right)\right]\left(\frac{d^2\varphi_2}{d\eta^2}-\frac{1+b}{\eta}\frac{d\varphi_2}{d\eta}-\varphi_2\right)=0 \tag{4.3}$$

The problem therefore reduces to seeking the function of the inhomogeneous equation

$$\frac{d^2\varphi_2}{d\eta^2}-\frac{1+b}{\eta}\frac{d\varphi_2}{d\eta}-\varphi_2=\varphi^+ \tag{4.4}$$

where φ^+ is the general solution of the equation

$$\frac{d^2\varphi^+}{d\eta^2}+\frac{1-b}{\eta}\frac{d\varphi^+}{d\eta}-\left(1+\frac{1+b}{\eta^2}\right)\varphi^+=0 \tag{4.5}$$

Solving (4.5) and (4.4), we find

$$\begin{aligned}\varphi_2 = z_1^{\,n}\{C_1 I_n(\alpha z_1)+C_2 K_n(\alpha z_1)+\\ +C_3\left[I_n(\alpha z_1)\int K_n^{\,2}(\alpha z_1)\,dz_1-K_n(\alpha z_1)\int I_n(\alpha z_1)K_n(\alpha z_1)\,dz_1\right]+\\ +C_4\left[I_n(\alpha z_1)\int I_n(\alpha z_1)K_n(\alpha z_1)\,dz_1-K_n(\alpha z_1)\int I_n^{\,2}(\alpha z_1)\,dz_1\right]\Big\},\quad n={}^1\!/_2(b+2)\end{aligned} \tag{4.6}$$

Here $I_n(\alpha z_1)$, $K_n(\alpha z_1)$ are Bessel functions of imaginary argument, of the first and second kind of order n, respectively. If the quantity $^1/_2\,(b+2)$ equals half of an odd number, then the Bessel functions, as is known, reduce to combinations of elementary functions. In this case it is easy to transform (4.6) successfully into tabulated functions, so that it is not difficult to obtain numerical results. For example, for $b=-1$ the solution (4.6) can be represented as

$$\begin{aligned}\varphi_2 = Ae^{\alpha z_1}+Be^{-\alpha z_1}+C\left[e^{\alpha z_1}\ln z_1-e^{-\alpha z_1}\operatorname{Ei}(2\alpha z_1)\right]+\\ +D\left[e^{-\alpha z_1}\ln z_1-e^{\alpha z_1}\operatorname{Ei}(-2\alpha z_1)\right]\end{aligned} \tag{4.7}$$

Here $\operatorname{Ei}(2\alpha z_1)$ and $\operatorname{Ei}(-2\alpha z_1)$ are exponential integral functions.

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